Electromagnetic Field Energy

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1 Problem

The (time-dependent) electromagnetic field energy in vacuum (or in a medium where $\mathbf{D} = \mathbf{E}$ and $\mathbf{B} = \mathbf{H}$) is given as

$$U_{EM} = \int \frac{E^2 + B^2}{8\pi} d\mathrm{Vol}, \tag{1}$$

in Gaussian units. Here, $\mathbf{D}$ is the electric displacement vector, $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic induction, and $\mathbf{H}$ is the magnetic field.

In static situations the electromagnetic energy can also be expressed in terms of sources and potentials as

$$U_{EM} = \frac{1}{2} \int \left(\rho\phi + \frac{\mathbf{J} \cdot \mathbf{A}}{c} \right) d\mathrm{Vol}, \tag{2}$$

where ρ is the charge density, $\mathbf{J}$ is the current density, ϕ is the scalar potential and $\mathbf{A}$ is the vector potential.

Time-dependent electromagnetic fields include radiation fields that effectively decouple from the sources. Verify that the expression (1) cannot in general be transformed into expression (2).

The Lorentz invariant quantity $E^2 - B^2$ vanishes for radiation fields. Hence the integral

$$\int (E^2 - B^2) d\mathrm{Vol} \tag{3}$$

excludes the contribution from the radiation fields, and remains related to the sources of the fields. Show that eq. (3) can be transformed into an integral of the invariant $(j \cdot A) = c\rho\phi - \mathbf{J} \cdot \mathbf{A}$ plus the time derivative of another integral.

The quantity $E^2 - B^2$ has the additional significance of being the Lagrangian density of the "free" electromagnetic field [1], while $\rho\phi - \mathbf{J} \cdot \mathbf{A}/c$ is also considered to be the interaction term in the Lagrangian between the field and sources. The above argument indicates that the "free" fields retain a kind of memory of their sources, as they must.

2 Solution

The aspects of this problem related to $E^2 - B^2$ were suggested by J.D. Jackson.

To bring the potentials into the field energy (1), we recall that since $\nabla \cdot \mathbf{B} = 0$ always, the magnetic induction can always be related to a vector potential $\mathbf{A}$ according to

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{4}$$

Faraday's law can therefore be written as

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}. \tag{5}$$

Thus, the curl of the quantity

$$\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \tag{6}$$

vanishes, which means that this quantity can be expressed as $-\nabla\phi$ where ϕ is a scalar potential. That is, the electric field can be written in terms of potentials as

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}. \tag{7}$$

Inserting expression (7) into the electric part of eq. (1), we find

$$
\begin{aligned}
U_E &= \frac{1}{8\pi}\int E^2\, d\text{Vol} = -\frac{1}{8\pi}\int \mathbf{E}\cdot\nabla\phi\, d\text{Vol} - \frac{1}{8\pi c}\int \mathbf{E}\cdot\frac{\partial \mathbf{A}}{\partial t}d\text{Vol} \\
&= -\frac{1}{8\pi}\int \nabla\cdot(\phi\mathbf{E})\, d\text{Vol} + \frac{1}{8\pi}\int \phi\nabla\cdot\mathbf{E}\, d\text{Vol} - \frac{1}{8\pi c}\int \epsilon\mathbf{E}\cdot\frac{\partial \mathbf{A}}{\partial t}d\text{Vol} \\
&= -\frac{1}{8\pi}\int \phi\mathbf{D}\cdot d\mathbf{S} + \frac{1}{2}\int \rho\phi\, d\text{Vol} - \frac{1}{8\pi c}\int \mathbf{E}\cdot\frac{\partial \mathbf{A}}{\partial t}d\text{Vol}
\end{aligned}
\tag{8}
$$

using the Maxwell equation,

$$\nabla\cdot\mathbf{E} = 4\pi\rho. \tag{9}$$

Likewise, the magnetic energy can be written

$$
\begin{aligned}
U_M &= \frac{1}{8\pi}\int B^2\, d\text{Vol} = \frac{1}{8\pi}\int \mathbf{B}\cdot\nabla\times\mathbf{A}\, d\text{Vol} \\
&= -\frac{1}{8\pi}\int \nabla\cdot\mathbf{B}\times\mathbf{A}\, d\text{Vol} + \frac{1}{8\pi}\int \mathbf{A}\cdot\nabla\times\mathbf{B}\, d\text{Vol} \\
&= -\frac{1}{8\pi}\int \mathbf{B}\times\mathbf{A}\cdot d\mathbf{S} + \frac{1}{2c}\int \mathbf{J}\cdot\mathbf{A}\, d\text{Vol} + \frac{1}{8\pi c}\int \mathbf{A}\cdot\frac{\partial \mathbf{E}}{\partial t}d\text{Vol},
\end{aligned}
\tag{10}
$$

using the Maxwell equation

$$\nabla\times\mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}. \tag{11}$$

For charges and current that occupy only a finite volume, the surface integrals in eqs. (8) and (10) go to zero as the surface becomes large. To see this, first suppose that there are no wave fields, so the potentials fall off at least as fast as $1/r$, and the quasistatic fields fall off at least as fast as $1/r^2$. Since the surface area increases as r^2, the surface integrals fall off as $1/r$ and can be neglected. If there are wave fields, then for large enough r, the leading terms of $\mathbf{A}$, $\mathbf{E}$ and $\mathbf{B}$ are all orthogonal to the surface vector $d\mathbf{S}$ and again the surface integrals can be neglected. An alternative justification for neglecting these integrals is given by Stratton [2].

Thus, the electromagnetic field energy can be written as

$$U_{EM} = U_E + U_M = \frac{1}{2}\int\left(\rho\phi + \frac{\mathbf{J}\cdot\mathbf{A}}{c}\right)d\text{Vol} + \frac{1}{8\pi c}\int\left(\mathbf{A}\cdot\frac{\partial \mathbf{E}}{\partial t} - \mathbf{E}\cdot\frac{\partial \mathbf{A}}{\partial t}\right)d\text{Vol}. \tag{12}$$

The second integral in eq. (12) can be transformed in various ways, but in general it is nonzero. The time-dependent electromagnetic energy (1) includes additional physical effects, the radiation fields, not accounted for in the expression (2).

The difference between the electric and the magnetic energies, in which contributions from radiation fields cancel, is

$$
\begin{aligned}
U_E - U_M &= \int \frac{E^2 - B^2}{8\pi} \, d\mathrm{Vol} \\
&= \frac{1}{2} \int \left(\rho\phi - \frac{\mathbf{J} \cdot \mathbf{A}}{c} \right) d\mathrm{Vol} - \frac{1}{8\pi c} \int \left(\mathbf{A} \cdot \frac{\partial \mathbf{E}}{\partial t} + \mathbf{E} \cdot \frac{\partial \mathbf{A}}{\partial t} \right) d\mathrm{Vol} \\
&= \frac{1}{2c} \int (j \cdot A) \, d\mathrm{Vol} - \frac{1}{8\pi c} \frac{d}{dt} \int \mathbf{A} \cdot \mathbf{E} \, d\mathrm{Vol},
\end{aligned}
\tag{13}
$$

where $j = (c\rho, \mathbf{J})$ is the 4-vector current density and $A = (\phi, \mathbf{A})$ is the 4-potential.

Since both $E^2 - B^2$ and $(j \cdot A)$ are Lorentz invariants, it should be possible to write $d\mathbf{A} \cdot \mathbf{E}/dt$ in an invariant form also. Indeed, that can be done using the covariant derivative

$$
\partial = \left(\frac{d}{cdt}, -\nabla \right),
\tag{14}
$$

and the time-like unit vector n whose components in the lab frame are

$$
n = (1, 0, 0, 0).
\tag{15}
$$

Then,

$$
\frac{d\mathbf{A} \cdot \mathbf{E}}{dt} = c(n \cdot \partial) \left[(n \cdot \partial) \left(\frac{A \cdot A}{2} \right) - (A \cdot \partial)(n \cdot A) \right],
\tag{16}
$$

for what it's worth.

3   References

[1] See, for example, J.D. Jackson, *Classical Electrodynamics*, 3rd ed. (Wiley, New York, 1998).

[2] J.A. Stratton, *Electromagnetic Theory* (McGraw-Hill, New York, 1941), p. 134.